Finding the Area between Two Curves

In this section we will explore how to find the area between two curves. I think that you will find the concept very easy, since we already know how to find the area between a curve and the x-axis. This concept is just a logical extension of the previous skill.

We know that to find the area between a curve and the x-axis we can take the integral of the function between our limits of integration. To find the area between two curves which intersect at the x-values of a and b, we will have to do some subtracting of integrals. Let’s look at a graph:

In this picture, the bright green area is the area between curves f (x) and g (x). The two curves intersect at x = a and b. If I integrate f (x) from a to b I will get the area which encompasses both green sections - everything between f (x) and the x-axis within the boundaries of x = a and x = b. If I integrate g (x) within the same limits of integration I will get the area shown by the dark green. If I want the area between the two curves, the light green area, then I will have to subtract the second integral from the first.

It is easy to see on the graph and there are no traps to snare you. It is really that straightforward as long as you know which function lies on top and where the two functions intersect. If the order of top and bottom function changes within the interval then you will have to split the problem up into separate sections.

Thus the steps for finding the area between two functions are:

a) Graph the functions, to see which is on top.
b) Find the points of intersections of the curves unless they are not needed because you have already been given the limits of integration. In the formula, they are a and b, the lower and upper limits of integration.
c) Thus the area between two curves is

\[ A = \int_{a}^{b} \text{(top function)} \, dx - \int_{a}^{b} \text{(bottom function)} \, dx \]

Since the limits of integration are the same for each integral, you can combine these to simply get:

\[ \int_{a}^{b} [f(x) - g(x)] \, dx \]

where \( f(x) \) is the top function and \( g(x) \) is the bottom function. **Doing this can sometimes greatly simplify the expression which you have to integrate and often save you a bunch of work, so it is a good habit to get into.**

d) Find and evaluate the integral. The number you get is the area between the curves. **You do not have to worry about areas lying below the x-axis.** Since you are subtracting the two integrals, it will always end up positive. Think of it: Big positive - smaller positive = positive. Negative - "bigger" negative = positive.
Finding the Area between Two Curves - Examples

The steps for finding the area between two functions are:

1 – Determine which function is on top
2 – Determine the points of intersection if they are not given to you
3 – Set up the integral: \[ \int_{a}^{b} [f(x) - g(x)] \, dx \] where \( f(x) \) is the upper function.
4 – Find the antiderivative and evaluate. Sometimes you will be told to evaluate the integral with your calculator instead of doing it the long way.

Now let's look at some examples.

A: Find the area between the graphs of \( f(x) = \ln 2x \) and \( g(x) = \ln x \) between \( x = 1 \) and \( 5 \).

If you graph these, you will see that \( f(x) \) always lies on the top within the interval. We do not have to find the intersection of the curves because we were given the limits of integration.

\[
A = \int_{1}^{5} [\ln(2x) - \ln x] \, dx
\]

Write the area integral.

\[
A = \int_{1}^{5} \ln \left( \frac{2x}{x} \right) \, dx = \int_{1}^{5} \ln 2 \, dx
\]

Now we want to simplify the expression within the integral using the laws of logs. Notice how the combining of the two integrals did wonders for our expression. Had we not done this we would have been hard-pressed to find the antiderivative with the methods we now have.

Now we take the antiderivative (remember that \( \ln 2 \) is just a constant).

\[
A = (\ln 2) [x]_{1}^{5}
\]

Evaluate.

\[
= (\ln 2) [5 - 1]
\]

or

\[
= 4 \ln 2
\]

\[
= \ln 2^4 = \ln 16
\]
B: Find the area between the graphs of \( f(t) = \frac{1}{2} \sec^2 t \) and \( g(t) = -4 \sin^2 t \) between \( x = -\pi/3 \) and \( \pi/3 \).

If you graph these, you will see that \( f(t) \) always lies on the top within the interval. We do not have to find the intersection of the curves because we were given the limits of integration.

\[
A = \int_{-\pi/3}^{\pi/3} \left[ \frac{1}{2} \sec^2 t + 4 \sin^2 t \right] dt
\]

We have to make a trig substitution in the second integral in order for us to take the integral. We will use the double angle formula that we have used before.

We can take the antiderivative now. I had to use substitution for the second part of the second integral (\( u = 2t \) and \( du = 2 \, dt \)).

\[
= \frac{1}{2} \left[ \tan \frac{\pi}{3} - \tan \left(-\frac{\pi}{3}\right) \right] + 2 \left[ \left(\frac{\pi}{3} - \frac{1}{2} \sin 2\pi/3\right) - \left(-\frac{\pi}{3} - \frac{1}{2} \sin -2\pi/3\right) \right]
\]

\[
= \frac{1}{2} \left[ \sqrt{3} - -\sqrt{3} \right] + 2 \left[ \frac{\pi}{3} - 1/2 \times (\sqrt{3}/2) + \pi/3 + 1/2 \times (-\sqrt{3}/2) \right]
\]

\[
= \sqrt{3} + 4\pi/3 - \sqrt{3} = 4\pi/3
\]

Evaluate.

Wow - what a tidy answer for such a long evaluation!

I will do some more examples in the HW examples document - make sure you understand how to do them before tackling the homework.
**Area between Two Curves with a Horizontal Orientation**

Sometimes it is necessary (or maybe just more convenient) to find the area between two curves with a horizontal orientation.

Let’s look at these two functions. We have to graph the functions here and find their points of intersections and determine their top/bottom orientation.

Now...this one is going to be tricky if we want to do it vertically because it is no easy task to tell which function is the top function when they are lying horizontally like this. But there is a way around this predicament. To set the stage, turn your paper counterclockwise, or if you are reading on your screen, turn you head sideways so that the graph looks like this:

We can see that the function that was on the right is now on top and the function that was on the left is now on the bottom. Cool! And because this happens, we can just integrate with a horizontal orientation. To do it, we integrate (right curve) - (left curve) and we integrate with respect to y instead of x:

**Area between curves with a Horizontal orientation:**

\[
A = \int_{a}^{b} [f(y) - g(y)] \, dy
\]

Where \( f(y) \) is the function on the right; \( g(y) \) is on the left.

All functions must be in terms of \( y \).
Let’s finish the problem, using this horizontal orientation. The functions are both already given to us in terms of $y$ (that means they are in the form $x = \ldots$).

\[3 - y^2 = \frac{1}{4} y^2\]
\[12 - 4y^2 = -y^2\]
\[12 = 3y^2\]
\[4 = y^2\]
\[y = \pm 2\]

I still need to know where these curves intersect so I will set the equations equal to each other and solve.

You can do this with your calculator if you like, but you should know how to do it algebraically too.

Since we are using a horizontal orientation, we must have $y$-limits. So we want to integrate from $y = -2$ to $y = 2$.

Combining the two integrals will help simplify things this time.

Now we take the antiderivative.

Evaluate.

As you can see, you will have to set the problems up carefully. Just remember that when you do this, you have to have everything in the integral in terms of $y$ - no stray $x$'s allowed! We use $y$-limits and equations in terms of $y$. 
** Homework Examples #1, 7 **

** On your homework I expect to see your sketches. ** You will not receive full credit without them. I also expect to see the ** work for figuring out the points of intersection. ** This is a good algebra review. On the test you will be able to use the calculator function to find these points of intersection if you know how to use it. Unfortunately I frequently find that calculus students forget the algebraic method of finding the points of intersection - and it is frequently needed on the non-calculator portion of the tests, so .... we will practice here.

** Another warning. ** Be very careful when sketching the regions. ** It may help to darken the boundary lines of the region that are given, ** and then double check to make sure the region you are looking at is, in fact, bounded on all sides by those lines. ** Remember, if you chose the wrong region, nothing will be right from then on! **

#1: ** Find the area of the shaded region analytically.

\[ y = \cos^2 x \]
\[ y = 1 \]
from \( x = 0 \) to \( \pi \)

\[ A = \int_{0}^{\pi} \left[ 1 - \cos^2 x \right] dx \]

\[ A = \int_{0}^{\pi} \sin^2 x \] \[ dx \]

\[ A = \int_{0}^{\pi} \left[ \frac{1}{2} - \frac{1}{2} \cos 2x \right] dx \]

\[ A = \left[ \frac{1}{2} x - \frac{1}{4} \sin 2x \right]_0^{\pi} \]

\[ A = \left( \frac{\pi}{2} - 0 \right) - (0 - 0) = \frac{\pi}{2} \]

These are our functions. The graph of the function is in the book; it looks like a bowl with the horizontal line on top and the curve on the bottom. They gave you the limits of integration. Set up the area integral. It will be the integral of the top function - bottom function. Simplify. We will have to use a basic trig identity and that power reduction formula again (page 580) before we can find the antiderivative. Find and evaluate the antiderivative. The second term requires the substitution \( u = 2x \), so \( \frac{1}{2} du = dx \).
#7: Find the area of the shaded region analytically.

I am going to do this one two ways, one with respect to $y$ (horizontally) and then again with respect to $x$ (vertically). The first method is easier and takes less work.

These are our functions. I can see the points of intersection on the graph, but I checked them to make sure. NEVER depend on the sketch - always do a double check. It is a common AP error!

If we set the problem up horizontally, then everything will be in terms of $y$, and we will then subtract the integral of the right - left functions.

Set up the integral. Make sure the limits come from the $y$-axis, not the $x$-axis.

Find and evaluate the antiderivative.
This time we want to set the problem up vertically, using a top and bottom function.

<table>
<thead>
<tr>
<th>Method Two:</th>
<th>This time we set the problem up vertically. Everything will be in terms of x, and we will then subtract the integral of the top - bottom functions. The problem here is that the top function is not always the same! To the left of $x = 1$, the line $y = x$ is on top. To the right of $x = 1$, the horizontal line is on top. That means we have to add two different integrals together to get the area.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Region 1:</td>
<td>Set up the integral.</td>
</tr>
<tr>
<td>Top function: $y = x$</td>
<td>Find and evaluate the antiderivative.</td>
</tr>
<tr>
<td>Bottom function: $y = \frac{1}{4}x^2$</td>
<td>There we go – same answer but a lot more work!</td>
</tr>
<tr>
<td>Limits: $x = 0$ to $1$</td>
<td>Being aware of the boundaries of your region is very important.</td>
</tr>
<tr>
<td>Region 2:</td>
<td></td>
</tr>
<tr>
<td>Top function: $y = 1$</td>
<td></td>
</tr>
<tr>
<td>Bottom function: $y = \frac{1}{4}x^2$</td>
<td></td>
</tr>
<tr>
<td>Limits: $x = 1$ to $2$</td>
<td></td>
</tr>
<tr>
<td>$A = \int_0^1 \left(x - \frac{1}{4}x^3\right) dx + \int_1^2 \left(1 - \frac{1}{4}x^2\right) dx$</td>
<td></td>
</tr>
<tr>
<td>$A = \left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_0^1 + \left[x - \frac{1}{12}x^3\right]_1^2$</td>
<td></td>
</tr>
<tr>
<td>$A = \left[\left(\frac{1}{2} - \frac{1}{12}\right) - 0\right] + \left[\left(2 - \frac{2}{3}\right) - \left(1 - \frac{1}{12}\right)\right]$</td>
<td></td>
</tr>
<tr>
<td>$A = \frac{5}{12} + \frac{4}{3} - \frac{11}{12} = \frac{10}{12} = \frac{5}{6}$</td>
<td></td>
</tr>
</tbody>
</table>
**Homework Examples #13, 21**

**On your homework I expect to see your sketches. You will not receive full credit without them. I also expect to see the work for figuring out the points of intersection.**  
**Another warning. Be very careful when sketching the regions. It may help to darken the boundary lines of the region that are given, and then double check to make sure the region you are looking at is, in fact, bounded on all sides by those lines. Remember, if you chose the wrong region, nothing will be right from then on!**

### #13:

Find the area of the region enclosed by the curves $y = 7 - 2x^2$ and $y = x^2 + 4$.

We are not given the points of intersection or the graph, so our first step is to graph the curves.

These are our functions. From the graph we can see that the downward facing parabola is always on top.

If you can't find the points of intersection easily through the graph, do it algebraically.

If you chose the wrong region, nothing will be right from then on!

Top function: $y = 7 - 2x^2$  
Bottom function: $y = x^2 + 4$

Limit of integration: $x = -1$ to $1$  
$7 - 2x^2 = x^2 + 4$  
$3x^2 = 3$  
$x = 1, -1$

Set up the area integral (top curve - bottom curve).

Simplify. I am also going to use symmetry to change the limits and double the integral. This will make our work easier in the evaluation of the antiderivative.

Find and evaluate the antiderivative.

$$A = 2 \int_0^1 (3 - 3x^2) \, dx = 2 \left[ 3x - x^3 \right]_0^1 = 4$$

$$A = 2 \left[ (3 - 1) - 0 \right] = 4$$
#21: Find the area of the region enclosed by the curves $x + y^2 = 0$ and $x + 3y^2 = 2$.

We are not given the points of intersection or the graph, so our first step is to graph the curves.

These are our functions. The parabolas are arranged horizontally, so we will set this up in terms of $y$. If we try to set it up vertically, the top function is not always the same and we will have to use more than one integral.

If you can't find the points of intersection easily through the graph, do it algebraically.

Set up the area integral (right curve - left curve).

Simplify. I am also going to use symmetry to change the limits and double the integral. This will make our work easier in the evaluation of the antiderivative.

Find and evaluate the antiderivative.