Integration by U-Substitution - the basics

Now let's look at a very common method of integration that will work on many integrals that cannot be simply done in our head. This is called integration by substitution, and we will follow a formal method of changing the variables. This works very well, works all the time, and is great for problems where we can't quite see the antiderivative. Whatever you do, do not avoid this method because you "don't want to write anything down" or think it is not as cool as doing it all in your head. It is a good method and most people have to use it from time to time no matter how good they are at integration. It also beats the heck out of getting problems wrong!

The problem with working in your head is that many people forget the reversal of the chain rule or get lost when doing the more complicated integrals. The solution is to use the substitution method of integration. In this method we will eliminate the embedded functions through a series of substitutions. These substitutions have to be picked out of thin air, but after practice it becomes fairly obvious what to use. In fact, we often have to use this method more than once when there is more than one embedded function involved.

There is one key rule to remember. Once you have made a substitution, such as \( u = g(x) \), there cannot be any stray \( x \)'s left in your integral. It is impossible to take the integral of a function with more than one variable in it!

Another note - I expect to see the substitution and all of its steps on your papers. Make sure that you know exactly what is happening. There is no magic here (other than the knack of picking the perfect substitution); it all is basic mathematical sense, so don't skip around. I want everyone to learn how to use this important method.

These are the basic steps in making a change of variables for integration by substitution. We will go through a lot of examples so that these steps will become clearer. Follow the method step by step and you will become an expert in using Substitution. This method is what you will use for about 80% of all the integrals you will see on the AP exam - learn it well.

1. Choose a substitution. Usually \( u = g(x) \), the inner function, such as a quantity in \((\quad)\) raised to a power or something under a radical sign.
2. Compute \( du = g'(x) \, dx \) (take the derivative, in differential form, of your chosen substitution \( u = g(x) \)).
3. Rewrite the integral in terms of the variable \( u \). Before you go further, make sure that you only have \( u \)'s in the integral. No stray \( x \)'s allowed.
4. Evaluate the integral in terms of \( u \).
5. Replace \( u \) by \( g(x) \) to obtain the antiderivative in terms of \( x \).
6. It is always a good idea to do a quick check by differentiating your answer to make sure you end up where you started.
Integration by U-Substitution and a Change of Variable

To review, these are the basic steps in making a change of variables for integration by substitution:

1. Choose a substitution. Usually $u = g(x)$, the inner function, such as a quantity raised to a power or something under a radical sign.
2. Compute $du = g'(x) dx$ (take the derivative, in differential form, of your chosen substitution $u = g(x)$).
3. Rewrite the integral in terms of the variable $u$. Before you go further, make sure that you only have $u$'s in the integral, no stray $x$'s allowed.
4. Evaluate the integral in terms of $u$.
5. Replace $u$ by $g(x)$ to obtain the antiderivative in terms of $x$.
6. It is always a good idea to do a quick check by differentiating your answer.

I am going to do some problems to illustrate this method.

A: \[ \int \sqrt{3 - 2x} \, dx \]

It should be pretty obvious that we will use the embedded function of $3 - 2x$. The embedded function is almost always the correct substitution to use!

<table>
<thead>
<tr>
<th>Let $u = 3 - 2x$</th>
<th>We are letting $u$ equal the embedded function.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$du = -2 , dx$</td>
<td>Now take the derivative, writing it in differential form. I originally had $du/dx = -2$.</td>
</tr>
<tr>
<td>$-\frac{1}{2} du = dx$</td>
<td>Solve for $dx$ because you have a $dx$ in the $\int$ which has to be gotten rid of (remember that we cannot mix variables in the integral).</td>
</tr>
</tbody>
</table>

Now we go back to the original $\int$ and substitute in the $u$ for $3 - 2x$ and $-\frac{1}{2}du$ for the $dx$. In other words we are going to get rid of all the $x$'s in the function, and in the process all the embedded functions will also disappear. The trick is to be sure you are doing mathematically correct stuff. Do not "force" the substitution. Only substitute what you actually have.

Continue:
\[ \int \sqrt{3 - 2x} \, dx = \int \sqrt{u} \left( -\frac{1}{2} \, du \right) \]

Take out the 3 - 2x and sub in the u.
Take out the dx and put in the -1/2 du.
Notice that there are no more x's in the integral. If you still have x's at this point, you must not proceed. You must stop and re-examine the substitution you chose.

\[ = -\frac{1}{2} \int u^{1/2} \, du \]

I moved the -1/2 out front and wrote the radical using exponents.

\[ = -\frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right] + C \]

Find the antiderivative by adding one to exponent and dividing by new exponent.

\[ = -\frac{1}{3} u^{3/2} + C \]

Simplify.

\[ = -\frac{1}{3} (3 - 2x)^{3/2} + C \]

Now you are done except for the final step - substitute the x's back into the answer. Since we let u = 3 - 2x in the first substitution, we now put a 3 - 2x back in place of the u.

Since this is an indefinite integral, we must put the +C at the end.

**B:** \[ \int 3y \sqrt{7 - 3y^2} \, dy \]

This problem is slightly more complicated in that we have y's in more than one place, so we have more choices for the substitution. However, a good rule of thumb is that your substitution should be the function which is the most deeply embedded. For this problem, this would be the stuff inside the square root sign.

But suppose that I did not realize this at first and I decided to use 3y as the substitution. Let's see what would happen:

<table>
<thead>
<tr>
<th>let u = 3y</th>
</tr>
</thead>
<tbody>
<tr>
<td>du = 3 dy</td>
</tr>
<tr>
<td>1/3 du = dy</td>
</tr>
</tbody>
</table>

We are ready to make the substitutions:

\[ \int 3y \sqrt{7 - 3y^2} \, dy \]

\[ \int u \sqrt{7 - 3 \left( \frac{1}{9} u^2 \right)} \left( \frac{1}{3} \, du \right) \]

\[ \int \frac{1}{3} u \sqrt{7 - \frac{1}{3} u^2} \, du \]

And this integral is still a mess to try and integrate! Ugh! And sometimes they will be impossible because you may still have y's left in the problem. As it was I had to do a bunch of figuring to find a substitution for the y^2 (y = 1/3 u so y^2 = 1/9u^2 ). Whenever stuff like this happens, try a
different substitution. Do not worry; everybody has to go back to square one with substitutions from time to time. Trying another is much better than trying to "force" it to work.

This time let's try the deeply embedded function of $7 - 3y^2$ for our substitution:

<table>
<thead>
<tr>
<th>let $u = 7 - 3y^2$</th>
<th>Write the substitution.</th>
</tr>
</thead>
<tbody>
<tr>
<td>then $du = -6y\ dy$</td>
<td>Take the derivative.</td>
</tr>
<tr>
<td>and $(-1/2)\ du = 3y\ dy$</td>
<td>Solve for what we have in the integrand.</td>
</tr>
</tbody>
</table>

Now let's sub these into our integral:

$$\int 3y\sqrt{7 - 3y^2}\ dy$$

We will be replacing the $7 - 3y^2$ by $u$ and the $3y\ dy$ by $(-1/2)\ du$.

$$\int \left(-\frac{1}{2}du\right)\sqrt{u} = -\frac{1}{2}\int u^{1/2}\ du$$

A much easier function to integrate! The correct substitution makes a BIG difference!

$$= -\frac{1}{2} \left[ \frac{2}{3} u^{3/2} \right] + C$$

Take the antiderivative.

$$= -\frac{1}{3}u^{3/2} + C$$

Simplify.

$$= -\frac{1}{3}\left(7 - 3y^2\right)^{3/2} + C$$

And finally, sub the $y$'s back in for $u$.

Once you get good at substitution, or maybe from the very first when the integrand is simple, you can go straight to the antiderivative by using pattern recognition. It requires that you recognize the result of using the chain rule to get a derivative. **Do not use this method unless you feel very comfortable with it!** Most errors come from students who want to "use the quick method" but don't really take the time to make sure they are doing it correctly. Use the change of variable if you have any doubt at all. It does not really take longer and you will get it right – much nicer than getting it wrong! Being able to find that embedded function, the $u$ or $g(x)$, is paramount in mastering integration by substitution and pattern recognition. Here are some more examples. I will go through the steps of substitution.

C) $$\int 2x(x^2 + 1)^4\ dx$$

<table>
<thead>
<tr>
<th>$\int 2x(x^2 + 1)^4\ dx$</th>
<th>Here is the integral.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = x^2 + 1$</td>
<td>Choose the $u$-substitution, find $du$, and match with the integrand.</td>
</tr>
<tr>
<td>$du = 2x\ dx$</td>
<td>Make the substitution.</td>
</tr>
</tbody>
</table>

$$\int u^4\ du$$

$$= \frac{1}{5}u^5 + C$$

Find the antiderivative.

$$= \frac{1}{5}(x^2 + 1)^5 + C$$

Reverse the substitution. Do a quick mental check by taking the derivative with the general power rule and making sure it matches the integrand you started with.
D) \[ \int 3x^2 \sqrt{x^3+1} \, dx . \]

<table>
<thead>
<tr>
<th>[ \int 3x^2 \sqrt{x^3+1} , dx ]</th>
<th>Here is the integral.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ u = x^3 + 1 ]</td>
<td>Choose the u-substitution, find ( du ), and match with the integrand.</td>
</tr>
<tr>
<td>[ du = 3x^2 , dx ]</td>
<td>Make the substitution.</td>
</tr>
<tr>
<td>[ \int u^{1/2} , du ]</td>
<td>Find the antiderivative.</td>
</tr>
<tr>
<td>[ \frac{2}{3} u^{3/2} + C ]</td>
<td>Reverse the substitution. Do a quick mental check by taking the derivative with the general power rule and making sure it matches the integrand you started with.</td>
</tr>
<tr>
<td>[ \frac{2}{3} (x^3 + 1)^{3/2} + C ]</td>
<td></td>
</tr>
</tbody>
</table>

E) \[ \int \sec^2 x \left( \tan x + 3 \right) \, dx . \]

<table>
<thead>
<tr>
<th>[ \int \sec^2 x \left( \tan x + 3 \right) , dx ]</th>
<th>Here is the integral.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ u = \tan x + 3 ]</td>
<td>Choose the u-substitution, find ( du ), and match with the integrand.</td>
</tr>
<tr>
<td>[ du = \sec^2 x , dx ]</td>
<td>Make the substitution.</td>
</tr>
<tr>
<td>[ \int u , du ]</td>
<td>Find the antiderivative.</td>
</tr>
<tr>
<td>[ \frac{1}{2} u^2 + C ]</td>
<td>Reverse the substitution. Do a quick mental check by taking the derivative with the general power rule and making sure it matches the integrand you started with.</td>
</tr>
<tr>
<td>[ \frac{1}{2} \left( \tan x + 3 \right)^2 + C ]</td>
<td></td>
</tr>
</tbody>
</table>
Integration by U-Substitution - Examples

Let's do a few more examples that are a bit more complicated.

The next three problems involve multiplying or dividing by a constant to make the pattern work. For some people this is easy (I think you have to be able to easily visualize functions and their derivatives in your mind for this to be easy), for others it seems like black magic. If you fall into the latter category, don't despair. The method of integration by substitution with a change of variable takes the guesswork out of it.

F) \[ \int x(x^2 + 1)^4 \, dx. \]

$$\int x(x^2 + 1)^4 \, dx$$

Here is the integral.

<table>
<thead>
<tr>
<th>( u = x^2 + 1 )</th>
<th>( du = 2x , dx )</th>
<th>( \frac{1}{2} , du = x , dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} \int u^4 , du )</td>
<td>( \frac{1}{2} \left[ \frac{1}{5} u^5 \right] + C )</td>
<td>( \frac{1}{10} (x^2 + 1)^5 + C )</td>
</tr>
</tbody>
</table>

Choose the u-substitution, find \( du \), and match with the integrand. This time, the derivative is not quite the same as the rest of the integrand; it is off by a factor of 2, so I move the 2 over with the \( du \) so that what I have on the right will exactly match what is left in the integrand.

Make the substitution.

Find the antiderivative.

Reverse the substitution. Do a quick mental check by taking the derivative with the general power rule and making sure it matches the integrand you started with.
G) \[ \int x^2\sqrt{x^3+1} \, dx . \]

\[
\begin{array}{|c|c|}
\hline
\text{Integral} & \text{Here is the integral.} \\
\hline
\int x^2\sqrt{x^3+1} \, dx & \text{Choose the } u\text{-substitution, find } du, \text{ and match with the integrand. This time, the derivative is not quite the same as the rest of the integrand; it is off by a factor of 3, so I move the } 3 \text{ over with the } du \text{ so that } \textbf{what I have on the right will exactly match what is left in the integrand}. \\
\ hline
u = x^3 + 1 & \text{Make the substitution.} \\
\ hline
\frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2 \, dx & \text{Find the antiderivative.} \\
(1/3) \, du = x^2 \, dx & \text{Reverse the substitution. Do a quick mental check by taking the derivative with the general power rule and making sure it matches the integrand you started with.} \\
\ hline
\frac{1}{3} \int u^{1/2} \, du & \\
\frac{1}{3} \left[ \frac{2}{3} u^{3/2} \right] + C & \\
\frac{2}{9} (x^3 + 1)^{3/2} + C & \\
\hline
\end{array}
\]

H) \[ \int 2 \sec^2 x (\tan x + 3) \, dx . \]

\[
\begin{array}{|c|c|}
\hline
\text{Integral} & \text{Here is the integral.} \\
\hline
\int 2 \sec^2 x (\tan x + 3) \, dx & \text{Choose the } u\text{-substitution, find } du, \text{ and match with the integrand. Again, the derivative is not quite the same as the rest of the integrand. This time I multiply both sides by 2 to make it match.} \\
\ hline
u = \tan x + 3 & \text{Make the substitution.} \\
\ hline
\frac{du}{dx} = \sec^2 x \Rightarrow du = \sec^2 x \, dx & \text{Find the antiderivative.} \\
2 \, du = 2\sec^2 x \, dx & \text{Reverse the substitution. Do a quick mental check by taking the derivative with the general power rule and making sure it matches the integrand you started with.} \\
\ hline
2 \int u \, du & \\
2 \left[ \frac{1}{2} u^2 \right] + C & \\
(\tan x + 3)^2 + C & \\
\hline
\end{array}
\]
Definite Integrals and Substitution

Now we have to look at what to do with definite integrals when using substitution. We know that when finding a definite integral, we first find the antiderivative and then evaluate by taking the value of the antiderivative at the upper limit of integration minus the value of the antiderivative at the lower limit of integration. The method is the same when we are using substitution, but you have to be careful about what variable you are using. We will have to do one of two things, either of which is just fine:

1) Change the limits of integration to fit the u’s at the same time you make the rest of the substitutions.
2) Treat the integral as an indefinite integral (ignore the limits of integration - leave them off), find the integral with your u’s, substitute the x’s back into the antiderivative and THEN evaluate, using the original limits of integration.

It does not matter which method you use, but if you choose method #2, leave off the limits of integration until after you have finished integrating with the u’s and reversed the substitution. I got a message from a previous year’s student who said that his calc teacher was very picky about the notation! No mixing of x’s and u’s allowed!

A: \( \int_{0}^{1} t^3 (1 + t^4)^3 \, dt \). Stop a minute and look at the limits of integration. Those are limits from \( t = 0 \) to \( t = 1 \). So whenever we have u’s in the problem, we will not be able to use those limits of integration. Be careful!

<table>
<thead>
<tr>
<th>Original problem.</th>
<th>( \int_{0}^{1} t^3 (1 + t^4)^3 , dt )</th>
</tr>
</thead>
</table>
| Find your substitutions. | \( u = 1 + t^4 \)
\( du = 4t^3 \, dt \)
\( \frac{1}{4} \, du = t^3 \, dt \) |
| Make the substitutions, ignoring the limits of integration. Leave them out. | \( \int u^3 \left( \frac{1}{4} \, du \right) \) |
| Simplify and take the antiderivative. | \( \frac{1}{4} \int u^3 \, du \)
\( = \frac{1}{4} \left( \frac{1}{4} u^4 \right) \)
\( = \frac{1}{16} u^4 \) |

Now evaluate the definite integral. As I said before, there are two methods. I am going to go through each one below. It does not matter which you use, as long as you remember not to mix your u’s and t’s.
Method I: We will evaluate the antiderivative with the u's still in there. However since the original limits of integration were for \( t = 0 \) to 1 and now we have u's in the problem we have to look back at the substitution we made.

We let \( u = 1 + t^4 \).

That means that when \( t = 0 \) (our original lower limit of integration), \( u = 1 + 0^4 = 1 \) and when \( t = 1 \) (the original upper limit of integration), \( u = 1 + 1^4 = 2 \). Hence our new limits of integration are \( u = 1 \) to 2.

Evaluate:

\[
\left[ \frac{1}{16} u^4 \right]_1^2
\]

Evaluate using the new limits of integration.

\[
\left( \frac{1}{16} \right) \left[ 2^4 - 1^4 \right]
\]

(value at \( u = 2 \)) - (value at \( u = 1 \))

\[
\left( \frac{1}{16} \right) \left[ 16 - 1 \right] = \frac{15}{16}
\]

Simplify and get the final answer.

Method II: We will evaluate the antiderivative after we have substituted the t's back into the problem. This way we can use the same limits of integration that we started with.

\[
\left[ \frac{1}{16} \left(1 + t^4\right)^4 \right]_0^1
\]

Put the t's back into the antiderivative and get ready to evaluate.

\[
\frac{1}{16} \left[ (1+1^4)^4 - (1+0^4)^4 \right]
\]

Evaluate using the original limits of integration.

(value at \( t = 1 \)) - (value at \( t = 0 \))

\[
\frac{1}{16} \left[ 16 - 1 \right] = \frac{15}{16}
\]

Simplify and get the final answer.

Same answer either way. Just stick to one method or the other, don't mix them within the same problem!

The nice thing about integration by substitution is that it will eliminate some of the errors that many make when taking the antiderivative but forgetting about reversing the chain rule. It takes care of it for you. Practice, practice, practice! This is an important method that we will be using for the rest of the year. **Substitution is the most important method of integration that there is.**
Differential Equations – Separation of Variables

A separable differential equation is one which is in the form $\frac{dy}{dx} = g(x) \cdot h(y)$. In other words it is an equation where your $\frac{dy}{dx}$ is equal to a function containing the product or quotient of a function of $x$ and a function of $y$. To take the integral we will have to separate the variables and put the $y$'s on one side with the $dy$ and the $x$'s on the other with the $dx$. Then you take the antiderivative of both sides. The only trick is to be sure to follow the rules of integration and to be sure to use substitution whenever it is needed.

Now I will do an example here. There will be more examples in the homework examples.

A:

\[
\frac{dy}{dx} = \frac{e^x(y-1)}{-2(e^x+4)}
\]

This is the differential equation. We can't take the integral yet because we have $x$’s and $y$’s mixed together on the right side of the equation.

\[
\int \frac{dy}{y-1} = \int \frac{e^x}{-2(e^x+4)} \, dx
\]

We will separate the variables by dividing by the $y - 1$ and multiplying by $dx$.

\[
\int \frac{dy}{y-1} = \int \frac{e^x}{-2(e^x+4)} \, dx
\]

Set up the integral.

On the right side:

\[
u = e^x + 4
\]

\[
du = e^x \, dx
\]

\[
-\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln |u|
\]

\[
\ln |y - 1| = -\frac{1}{2} \ln (e^x + 4) + C
\]

Now we take the integrals.

I have to use substitution on the right side of the equation.

You can stop here! I will simplify some more but unless your directions say to find $y = f(x)$, you do not have to solve for $y$.

\[
2 \ln |y - 1| + \ln (e^x + 4) = C
\]

This can actually be simplified some more. I moved the right side to the left and multiplied by 2. $2C$ is the same as $C$ since it is just some constant.

\[
\ln (y - 1)^2 (e^x + 4) = C
\]

Use the laws of exponents to simplify.

\[
e^C = (y - 1)^2 (e^x + 4) \quad \text{or} \quad C = (y - 1)(e^x + 4)
\]

Now rewrite this in exponential form. $e^C$ is just a constant that we can rename again.

I think that the trickiest part of this section will be to find all of the antiderivatives without making silly mistakes. The concept of separating the variables is probably not too difficult. So ... be careful before taking any integral and make sure you are following the proper procedures: substitution, etc.
#11: Use substitution to evaluate the integral.

\[
\int \sqrt{\tan x \sec^2 x} \, dx
\]

This is the integral.

\[
u = \tan x \\
du = \sec^2 x \, dx
\]

This time you have to find the substitution. If you make an error, don't worry, just try something else! I always try the most deeply embedded function, and check the derivative of it in my head to see if I will find it elsewhere in the integrand.

\[
\int u^{1/2} \, du \\
\frac{2}{3} u^{3/2} + C \\
\frac{2}{3} \left( \tan x \right)^{3/2} + C
\]

Make the substitutions, find the antiderivative, and then reverse the substitution.

Do a quick mental check with the power rule to make sure you did it correctly.

#13: Use substitution to evaluate the integral.

\[
\int_6^x \frac{dx}{x \ln x}
\]

This is the integral.

\[
u = \ln x \\
du = \frac{1}{x} \, dx
\]

This time you have to find the substitution

\[
\int \frac{1}{u} \, du \\
\left[ \ln(u) \right]_6^x \\
\left[ \ln(\ln 6) \right] - \ln(\ln e) \\
\ln(\ln 6) - \ln(1) \\
\ln(\ln 6) - 0 = \ln(\ln 6)
\]

Make the substitutions, find the antiderivative, and then reverse the substitution. **Leave off the limits of integration because they deal with x's, not u's!!**

Reverse the substitution and then evaluate the definite integral.

There was a bit of simplification that needed to be done here.
#17: Use substitution to evaluate the integral.

<table>
<thead>
<tr>
<th>Integral</th>
<th>This is the integral.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int \frac{\ln^6 x}{x} , dx$</td>
<td>It may help to rewrite the integrand.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ReWrite</th>
<th>This time you have to find the substitution.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = \ln x$</td>
<td></td>
</tr>
<tr>
<td>$du = \frac{1}{x} , dx$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Substitution</th>
<th>Make the substitutions, find the antiderivative, and then reverse the substitution.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int u^6 , du$</td>
<td>Do a quick mental check with the power rule to make sure you did it correctly.</td>
</tr>
<tr>
<td>$\frac{1}{7} u^7 + C$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{7} (\ln x)^7 + C$</td>
<td></td>
</tr>
</tbody>
</table>
#21: Use substitution to evaluate the integral.

<table>
<thead>
<tr>
<th>[ \int \frac{\sin(2t + 1)}{\cos^2(2t + 1)} , dt ]</th>
<th>This is the integral.</th>
</tr>
</thead>
</table>
| \[ u = \cos(2t + 1) \]  
\[ du = -2\sin(2t + 1) \, dt \]  
\[ \frac{1}{2} \, du = \sin(2t + 1) \, dt \] | This time you have to find the substitution. Because the derivative of sine is not cosine squared, that will not be the correct substitution. |
| \[ -\frac{1}{2} \int \frac{1}{u^2} \, du = -\frac{1}{2} \int u^{-2} \, du \]  
\[ \frac{1}{2} \left[ -u^{-1} \right] + C \]  
\[ \frac{1}{2} \left( \cos(2t + 1) \right)^{-1} + C \] | Make the substitutions, find the antiderivative, and then reverse the substitution. You had to rewrite the integrand with negative exponents on this one. |
|  | Do a quick mental check with the power rule to make sure you did it correctly. |

#25: Use substitution to evaluate the integral.

<table>
<thead>
<tr>
<th>[ \int_1^3 \frac{x , dx}{x^2 + 1} ]</th>
<th>This is the integral.</th>
</tr>
</thead>
</table>
| \[ u = x^2 + 1 \]  
\[ du = 2x \, dx \]  
\[ \frac{1}{2} \, du = x \, dx \] | This time you have to find the substitution. |
| \[ \frac{1}{2} \int \frac{1}{u} \, du \]  
\[ \frac{1}{2} \left[ \ln u \right] \]  
\[ \frac{1}{2} \left[ \ln \left( x^2 + 1 \right) \right]_1^3 \]  
\[ \frac{1}{2} [\ln 10 - \ln 2] \]  
\[ \frac{1}{2} \ln 5 \] | Make the substitutions, find the antiderivative, and then reverse the substitution. Leave off the limits of integration because they deal with x's, not u's!!!! |
|  | Reverse the substitution and then evaluate the definite integral. There was a bit of simplification that should be done here. |
#3: Use the indicated substitution to evaluate the integral. Confirm the answer through differentiation.

\[ \int \sec 2x \tan 2x \, dx \]

<table>
<thead>
<tr>
<th>This is the integral.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ u = 2x ]</td>
</tr>
<tr>
<td>[ du = 2 , dx ]</td>
</tr>
<tr>
<td>( \frac{1}{2} , du = , dx )</td>
</tr>
</tbody>
</table>

They gave us the substitution that they wanted us to use.

\[ \frac{1}{2} \int \sec u \tan u \, du \]

Make the substitutions, find the antiderivative, and then reverse the substitution.

\[ \frac{1}{2} \sec u + C \]

\[ \frac{1}{2} \sec 2x + C \]

Check:

\[ \frac{d}{dx} \left( \frac{1}{2} \sec 2x + C \right) = \frac{1}{2} \left[ 2 \sec 2x + \tan 2x \right] \]

\[ = \sec 2x + \tan 2x \]

Because they asked you to, we will check the answer by taking its derivative with the chain rule.
**#7:** Use the indicated substitution to evaluate the integral. Confirm the answer through differentiation.

\[
\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} \, dt
\]

This is the integral.

\[
u = 1 - \cos \frac{t}{2}
\]
\[
du = \frac{1}{2} \sin \frac{t}{2} \, dt
\]
\[
2 \, du = \sin \frac{t}{2} \, dt
\]

They gave us the substitution that they wanted us to use.

\[
2 \int u^2 \, du
\]
\[
= \frac{2}{3} u^3 + C
\]
\[
= \frac{2}{3} \left(1 - \cos \frac{t}{2}\right)^3 + C
\]

Make the substitutions, find the antiderivative, and then reverse the substitution.

Check:

\[
\frac{d}{dx} \left[ \frac{2}{3} \left(1 - \cos \frac{t}{2}\right)^3 + C \right] = \frac{2}{3} \left(1 - \cos \frac{t}{2}\right)^2 \left(\frac{1}{2}\cos \frac{t}{2}\right)
\]

Because they asked you to, we will check the answer by taking its derivative with the general power rule.
### #31: Use a u-substitution and then evaluate the integral from u(a) to u(b).

\[
\int_{0}^{3} \sqrt{y+1} \, dy
\]

This is the integral.

<table>
<thead>
<tr>
<th>( u = y + 1 )</th>
<th>Find the substitution. <strong>This time they want us to change the limits too.</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( du = dy )</td>
<td></td>
</tr>
</tbody>
</table>

Limits:
- If \( y = 0 \), \( u = 1 \)
- If \( y = 3 \), \( u = 4 \)

\[
\int_{1}^{4} u^{1/2} \, du
\]

Make the substitutions, find the antiderivative.

\[
\frac{2}{3} \left[ u^{3/2} \right]_{1}^{4}
\]

\[
= \frac{2}{3} \left[ 8 - 1 \right] = \frac{14}{3}
\]

Since we changed the limits to match the u's, we do not have to reverse the substitution. Just evaluate the antiderivative with the new limits.
### #35: Use a u-substitution and then evaluate the integral from u(a) to u(b).

<table>
<thead>
<tr>
<th>Integral</th>
<th>This is the integral.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_{0}^{1} \frac{10\sqrt{\theta}}{(1 + \theta^{3/2})^2} , d\theta$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Substitution</th>
<th>Find the substitution. This time they want us to change the limits too.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 1 + \theta^{3/2}$</td>
<td></td>
</tr>
<tr>
<td>$du = \frac{3}{2} \theta^{1/2} , d\theta$</td>
<td></td>
</tr>
<tr>
<td>$\frac{2}{3} du = \theta^{1/2} , d\theta$</td>
<td></td>
</tr>
</tbody>
</table>

**Limits:**
- If $\theta = 0$, $u = 1$
- If $\theta = 1$, $u = 2$

<table>
<thead>
<tr>
<th>Antiderivative</th>
<th>Make the substitutions (I moved the 10 out front too), find the antiderivative.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{20}{3} \left[ \frac{1}{u^2} \right]_1^2$</td>
<td>Since we changed the limits to match the u's, we do not have to reverse the substitution. Just evaluate the antiderivative with the new limits.</td>
</tr>
<tr>
<td>$\frac{20}{3} \left[ -u^{-1} \right]_1^2$</td>
<td></td>
</tr>
<tr>
<td>$\frac{20}{3} \left[ \frac{1}{2} - (-1) \right] = \frac{10}{3}$</td>
<td></td>
</tr>
</tbody>
</table>
Homework Examples #39, 41, 43

#39: Solve the differential equation by separation of variables.

\[
\frac{dy}{dx} = (y + 5)(x + 2)
\]

This is the differential equation.

\[
\frac{1}{y + 5} \, dy = (x + 2) \, dx
\]

Separate the variables. Put the x’s on the right and the y’s on the left.

\[
\ln |y + 5| = \frac{1}{2} x^2 + 2x + C
\]

Integrate both sides of the equation.

\[
y + 5 = e^{\frac{1}{2}x^2+2x+C}
\]

This simplification is something we will do a lot of, so make sure you understand the steps.

\[
y + 5 = e^C \cdot e^{\frac{1}{2}x^2+2x}
\]

Change to exponential form.

\[
y + 5 = e^C \cdot e^{\frac{1}{2}x^2+2x}
\]

Simplify the right side with the rules of logs.

\[
y = Ce^{\frac{1}{2}x^2+2x} - 5
\]

Solve for y. Also, change the e^C to an arbitrary constant (it is just a constant, so we can change its name).

#41: Solve the differential equation by separation of variables.

\[
\frac{dy}{dx} = (\cos x) e^{y+\sin x}
\]

This is the differential equation.

\[
\frac{dy}{dx} = (\cos x) e^{y} e^{\sin x}
\]

Separate the variables. First I had to apply the rules of logs to the right side.

\[
e^{-y} dy = (\cos x) e^{\sin x}
\]

On the right side:

\[
\begin{align*}
  u &= \sin x \\
  du &= \cos x \, dx \\
  \int e^u \, du &= e^u \\
  -e^{-y} &= e^{\sin x} + C
\end{align*}
\]

Integrate both sides of the equation. I had to use substitution on the right side.
#43: Solve the initial value problem by separation of variables.

\[
\frac{dy}{dx} = -2xy^2 \\
y(1) = 0.25
\]

This is the differential equation and our initial conditions.

\[
y^{-2}dy = -2xdx
\]

Separate the variables.

\[
-\frac{1}{y} = -x^2 + C
\]

Integrate both sides of the equation.

\[
y(1) = 0.25
\]

\[
-\left(\frac{1}{4}\right) = -1 + C
\]

\[
C = -3
\]

I am going to solve for \( C \) now.

\[
-\frac{1}{y} = -x^2 - 3
\]

\[
y^{-1} = x^2 + 3
\]

Give the final answer. There is no real reason to simplify further because they did not ask to find \( y = f(x) \). If they had, we would have to solve for \( y \).

If we had, this would be the answer:

\[
y = \frac{1}{x^2 + 3}
\]